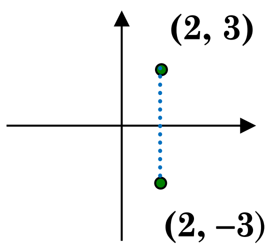
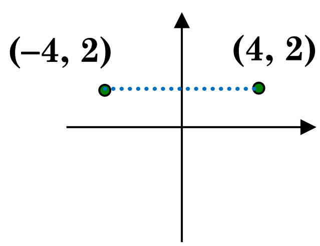
SYMMETRY – EVEN AND ODD FUNCTIONS

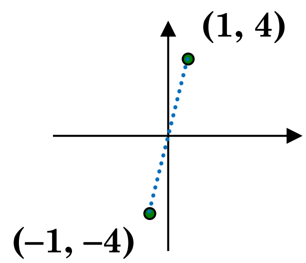
❑ *SYMMETRY OF A PAIR OF POINTS*



***x*-axis** symmetry



***y*-axis** symmetry



origin symmetry

In the first graph, notice that (2, 3) is directly above (2, –3), and that the two points are equidistant (equally distant) from the x-axis; that is, the dotted line is bisected by the x-axis. It's as though the bottom point is a reflection in the water of the top point. Also, note that the x-coordinates of the two points are the same and their y-coordinates are opposites. The two points are ***symmetric with respect to the x-axis***.

In the second graph, we see that (4, 2) is directly to the right of (–4, 2), and that the two points are equidistant from the y-axis; that is, the dotted line is bisected by the y-axis. It's like each point is a mirror image of the other. The two points are ***symmetric with respect to the y-axis.*** Also, can you see that the x-coordinates of the two points are opposites, and their y-coordinates are the same?

In the third graph, note that the two points are equidistant from the origin, and the dotted line is bisected by the origin. The two points are ***symmetric with respect to the origin***. Lastly, it should be clear that the x-coordinates of the two points are opposites, and their y-coordinates are also opposites.

Homework

1. For each of the following points, find the point that has ***x*-axis** symmetry with it:

   a. $(2, 7)$ b. $(-3, 4)$ c. $(5, \pi)$ d. $(4, 0)$ e. $(0, -3)$

2. For each of the following points, find the point that has ***y*-axis** symmetry with it:

   a. $(-2, 5)$ b. $(-1, -2)$ c. $(7, 0)$ d. $(\pi, -\sqrt{2})$ e. $(0, -2)$

3. For each of the following points, find the point that has **origin** symmetry with it:

   a. $(-3, 5)$ b. $(8, -7)$ c. $(0, 0)$ d. $(6, 0)$ e. $(0, -\pi)$

4. The line segment connecting the points P and Q is bisected by the y-axis. Does this imply that P and Q must be symmetric?

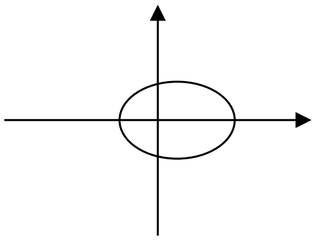
5. Let P be a point in the first quadrant and let Q be a point such that P and Q have x-axis symmetry. Then Q must lie in which quadrant?

6. Let (x, y) be a point in the third quadrant. Name the point such that the pair of points has origin symmetry.

❑ *SYMMETRY OF A GRAPH*

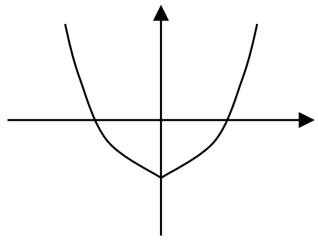
In the previous section, we discussed what it meant for a pair of points to have some kind of symmetry. Now we extend this concept for any graph in the plane.

***x*-axis symmetry**



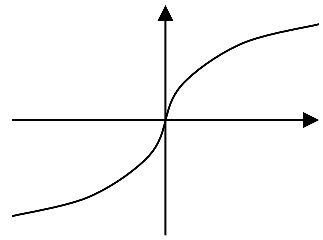
Given any point on the graph, there's an equidistant point on the other side of the *x*-axis.

***y*-axis symmetry**



Given any point on the graph, there's an equidistant point on the other side of the *y*-axis.

origin symmetry



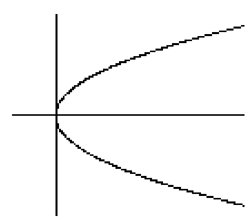
Given any point on the graph, there's an equidistant point on the other side of the origin.

The Litmus Tests for Symmetry

Now that we can visualize the three types of symmetry, we need a way to test a given equation for symmetry without having to graph it.

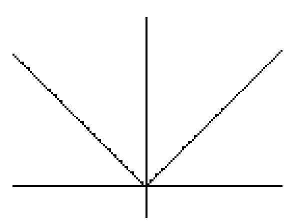
x-axis Replace $\boldsymbol{y}$ with $\boldsymbol{-y}$ everywhere in the given equation. If the new equation is equivalent to the old one, then the graph of the equation has ***x*-axis symmetry**.

For example, consider the sideways-opening parabola $x = y^2$. Replacing y with $-y$ gives the equation $x = (-y)^2$, which is the same as $x = y^2$. Since the replacement didn't really change the equation, we have x-axis symmetry.



y-axis Replace $\boldsymbol{x}$ with $\boldsymbol{-x}$ everywhere in the given equation. If the new equation is equivalent to the old one, then the graph of the equation has ***y*-axis symmetry**.

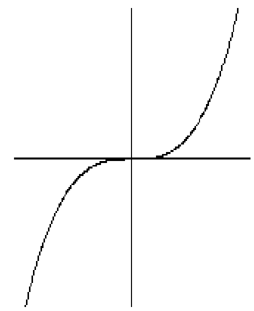
For instance, look at the absolute value function $y = |x|$. Replacing x with $-x$ gives the equation $y = |-x|$, which is actually the same



as $y = |x|$ (since the absolute values of a number and its opposite are the same). Thus, the graph has y-axis symmetry.

origin Replace $\boldsymbol{x}$ with $\boldsymbol{-x}$ and replace $\boldsymbol{y}$ with $\boldsymbol{-y}$ everywhere in the equation. If the new equation is equivalent to the old one, then the graph of the equation has **origin symmetry**.

Consider the cubic function $y = x^3$. Replacing x with $-x$ and y with $-y$ gives the equation $-y = (-x)^3$, which implies that $-y = -x^3$, which implies that $y = x^3$, so there was no net change in the equation. Therefore, the graph has origin symmetry.



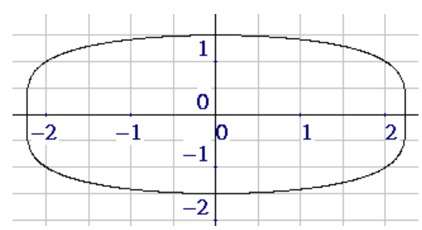
EXAMPLE 1: **Find all the symmetries of $\boldsymbol{x^2 + y^4 = 5}$.**

Solution:

x-axis Replace y with $-y$: $x^2 + (-y)^4 = 5 \Rightarrow x^2 + y^4 = 5$, the same as the original equation; therefore the graph has x-axis symmetry.

y-axis Replacing x with $-x$ will again result in the same equation; thus the graph has y-axis symmetry.

origin Replace both variables with their negatives: $(-x)^2 + (-y)^4 = 5 \Rightarrow x^2 + y^4 = 5$, the original equation; we conclude that the graph has origin symmetry as well.



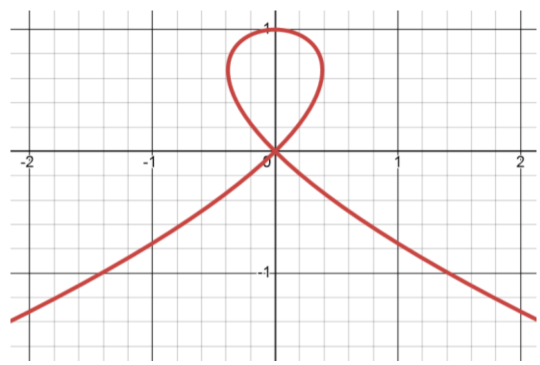
EXAMPLE 2: **Find all the symmetries of $y^2 = x^2 + y^3$.**

Solution:

x-axis Replace y with $-y$: $(-y)^2 = x^2 + (-y)^3$
$\Rightarrow y^2 = x^2 - y^3$, which is not the same as the original equation, nor can it be made to be the same. Thus, the graph does not have x-axis symmetry.

y-axis Replace x with $-x$: $y^2 = (-x)^2 + y^3$
$\Rightarrow y^2 = x^2 + y^3$, which is the same as the original equation. Therefore, the graph has y-axis symmetry.

origin Replace x with $-x$ and y with $-y$: $(-y)^2 = (-x)^2 + (-y)^3$
$\Rightarrow y^2 = x^2 - y^3$, which is not the same as the original equation — and will never be — no matter what we do to each side of the equation. We conclude that the graph does not have origin symmetry.



In summary, the graph has y-axis symmetry only.

Homework

7. Sketch a line with x-axis symmetry, a line with y-axis symmetry, and a line with origin symmetry.

8. a. Sketch a parabola with y-axis symmetry.
   b. Sketch a parabola with x-axis symmetry.

9. Sketch a circle that has x-axis symmetry, but neither of the other two.

10. Sketch a circle that has y-axis symmetry, but neither of the other two.

11. Sketch a circle that has origin symmetry, but neither of the other two.

12. A graph with origin symmetry contains some points in the 2nd quadrant. Which other quadrant <u>must</u> contain some points of the graph?

13. Use the litmus tests to find the symmetries of each graph:

    a. $x^2 - y^2 = 1$ b. $y = x^2 + x - 1$

    c. $y = x^2 - 8$ d. $x = |y|$

    e. $y = \dfrac{1}{x}$ f. $y = x^6 - x^4 + x^2$

14. If a graph has x-axis symmetry <u>and</u> y-axis symmetry, does that imply that it has origin symmetry? What about the converse? [See Chapter 1, *Logic*.]

❑ *EVEN AND ODD FUNCTIONS*

We end this chapter by categorizing functions based upon their symmetries.

If a function has y-axis symmetry, we call it an **even function**.

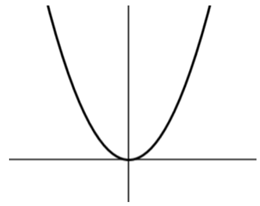
If a function has origin symmetry, we call it an **odd function**.

That's it — just two new terms that refer to the symmetries we already know.

Where do these two new terms come from?

The classic function with ***y*-axis** symmetry is our friend the parabola:

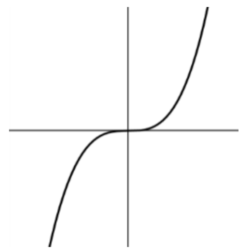
$y = x^2$



Look at the exponent of the function; it's an <u>even</u> number.

The classic function with **origin** symmetry is the cubic function:

$y = x^3$



Look at the exponent of the function; it's an <u>odd</u> number.

<u>Final Comment:</u> Note that if a graph has x-axis symmetry, it's NOT a function at all, so it certainly can't be an even or odd function.

Solutions

1. a. (2, –7) b. (–3, –4) c. (5, –π) d. (4, 0) e. (0, 3)

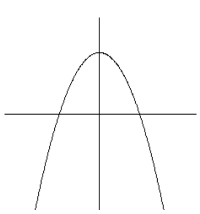
2. a. (2, 5) b. (1, –2) c. (–7, 0) d. $(-\pi, -\sqrt{2})$ e. (0, –2)

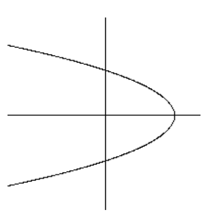
3. a. (3, –5) b. (–8, 7) c. (0, 0) d. (–6, 0) e. (0, π)

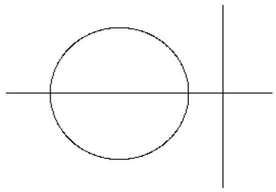
4. No **5.** IV **6.** $(-x, -y)$

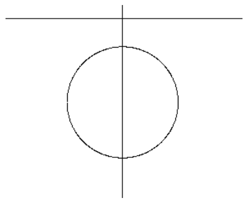
7. any vertical line; any horizontal line; any line through the origin

8.





9. 

10. 

11. Impossible **12**. Quadrant IV

13. a. all b. none c. y-axis d. x-axis e. origin f. y-axis

14. Yes; but the converse is false — consider $y = 2x$. It has origin symmetry but neither of the other two.

"Education is simply the soul of a society as it passes from one generation to another."

– **Gilbert Chesterton**